Announcements

1) If you lost .5 points on the last question on HW #2 for not doing a "converse", let me know.

Householder Triangulation

This is what Matlab is doing!

Compare the triangular matrices obtained by the "housel" algorithm from the book with Matlab's qr "command".

Only one catch - you need $j=1$ to $n-1$ instead of $j=1$ to n.

Cost

Is the cost of using Householder reflections to compute a QR decomposition any better than using Classical or modified Gram-Schmidt?

Cost by Picture

Let's do modified Gram-Schmidt first.

The algorithm has a row index $1 \le j \le m$, a column index $1 \le i \le n$, and an "outer loop" index.

Inner loop:

for $j = i+1$ to n

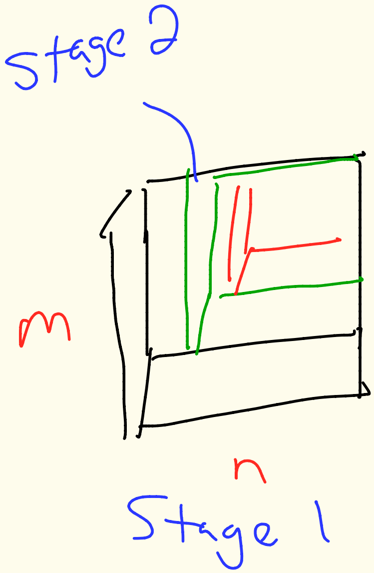
$$r_{i,j} = q_i^* v_j$$

$$v_j = v_j - r_{i,j} q_i$$

all these vectors are in $\mathbb{C}^m$

Approximately $4n$ flops, or 4 flops per column vector.

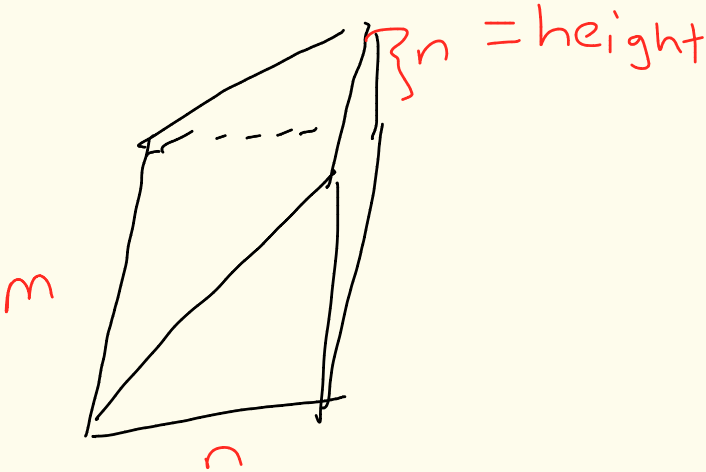
Operation count by picture:



height = outer loop index = n

each stage = 1 less column (inner loop index)

As m and n become larger, the figure looks more like:



right triangular prism

$$\text{Volume} = m \cdot (\text{area of a slice})$$

$$= m\left(\tfrac{1}{2} \cdot n^2\right)$$

$$= \frac{mn^2}{2}.$$

multiply by the number of operations per column (4) to get

$$\boxed{\text{cost} = 2mn^2}$$

Cost of Householder

"The work ... is dominated by the innermost loop"

$$A_{k:m,k:n} - 2v_k\left(v_k^* \cdot A_{k:m,k:n}\right)$$

Fixing an n, this is

$(m-k+1)$ multiplications

for $v_k^* \cdot A_{k:m,n}$,

then $2m$ multiplications

for $2 \cdot v_k (v_k^* A_{k:m,n})$,

and finally, m subtractions

for $A_{k:m,n} - 2v_k(v_k^* \cdot A_{k:m,n})$

This is then

$$(m-k+1) + 3m$$

$$= 4m - k + 1$$

flops, then multiply times the number of columns = always shrinking! At a given stage i, it is multiplication by $n-i+1$.

We get

$$(n-i+1)(4m-k+1)$$

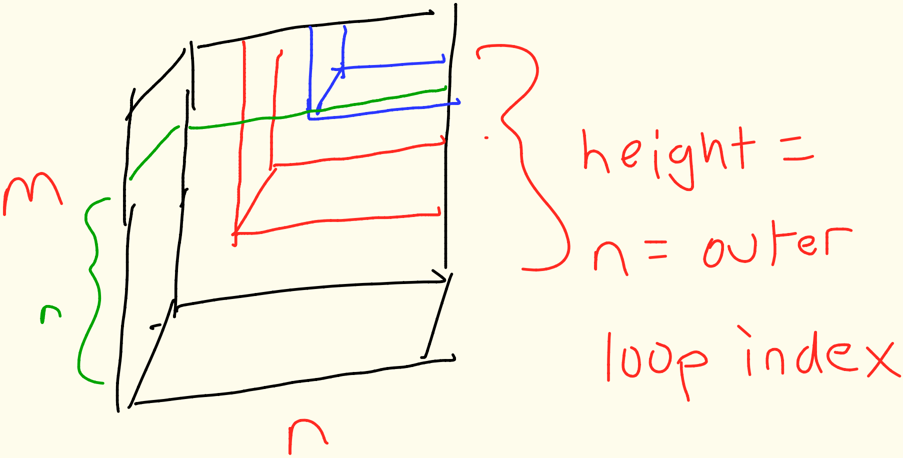
Then run through the indices.

Books picture:

More or less 4 flops per column element.

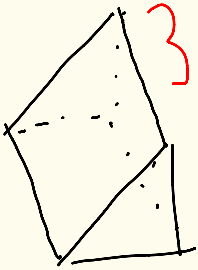
Both row and column index shrink by 1 at each stage.

Picture



Book splits up into two figures: Chops off the row index at n to get (in the limit)

$(m-n)$



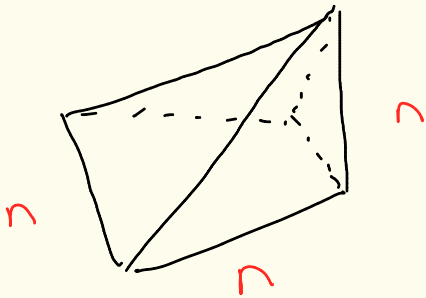
height n

triangular prism

n

Volume $\left(\frac{m-n}{2}\right) \cdot n^2$

and also



$$\text{Volume} = \frac{1}{3} n^3$$

Add the two to get a cost of

$$\left(\tfrac{1}{3}n^3 + \frac{(m-n)\cdot n^2}{2}\right)\cdot 4$$

$$= \left(\tfrac{1}{3}n^3 - \frac{n^3}{2} + \frac{mn^2}{2}\right)\cdot 4$$

$$= \left(\frac{mn^2}{2} - \frac{n^3}{6}\right)\cdot 4$$

$$= \boxed{2mn^2 - \frac{2n^3}{3}}$$

Least-Squares

or

Solving $Ax=b$

If $A \in \mathbb{C}^{m \times m}$ is invertible . . .

Example 1:

$$A = \begin{bmatrix} 2 & -6 \\ 1 & -3 \end{bmatrix}$$

$$b = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

Picture

Lemma: $\left(\ker(A^*) = (\operatorname{ran}(A))^\perp \right)$

Theorem: (least squares)

Definition : (pseudo inverse)

Application

Polynomial Interpolation

Example 2: (best-fit line)

QR Decomposition and Least-Squares

"modern-classical"

$$\sum_{k=1}^{n}\Big(\big(2(m-k+1)+2\big) + (m-k+1) + (n-k+1) \cdot \big((n-k+1)+(m-k+1) + (m-k)+(n-k+1)\big)\Big)$$

$$(n-k+1)(4m-4k+3)$$

$$= 4mn - 4kn + 3n - 4kn + 4k^2 - 3k + 4m - 4k + 3$$

$$\sum_{i=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$